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# Differential Sandwich Theorems for Multivalent Analytic Functions Associated with the Dziok-Srivastava Operator *

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**Abstract**

Differential subordination and superordination results are obtained for multivalent analytic functions in the open unit disk that are associated with the Dziok-Srivastava linear operator. These results are obtained by investigating appropriate classes of admissible functions. Sandwich-type results are also obtained.

**Keywords and Phrases:** *Hypergeometric function, Subordination, Superordination, Dziok-Srivastava linear operator, Convolution.*

# 1. Introduction

Let $\mathcal{H}(U)$ be the class of functions analytic in $U := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$, with $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H} \equiv \mathcal{H}[1, 1]$. Let $\mathcal{A}_n$ denote the class of all analytic functions of the form

$$f(z) = z^n + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \geq 1) \tag{1.1}$$

in $U$, and let $\mathcal{A}_1 := \mathcal{A}$. Let $f$ and $F$ be members of $\mathcal{H}(U)$. A function $f$ is said to be *subordinate* to $F$, or $F$ is said to be *superordinate* to $f$, if there exists a function $w$ analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = F(w(z))$. In such a case we write $f(z) \prec F(z)$. If $F$ is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$. For two functions $f, g \in \mathcal{A}_n$, where $f$ given by (1.1) and $g(z) = z^n + \sum_{k=n+1}^{\infty} b_k z^k$, the Hadamard product (or convolution) of $f$ and $g$ is defined by the series

$$(f * g)(z) := z^n + \sum_{k=n+1}^{\infty} a_k b_k z^k =: (g * f)(z).$$

For $\alpha_j \in \mathbb{C} \quad (j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leq m+1; l, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in \mathbb{N} := \{1, 2, 3\ldots\}). \end{cases}$$

Corresponding to the function

$$h_n(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z^n \; {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the Dziok-Srivastava operator [22] $\widetilde{H}_n^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \mathcal{A}_n \to \mathcal{A}_n$ is defined by the Hadamard product

$$\begin{aligned} &\widetilde{H}_n^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) \\ &:= h_n(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ &= z^n + \sum_{k=n+1}^{\infty} \frac{(\alpha_1)_{k-n} \ldots (\alpha_l)_{k-n}}{(\beta_1)_{k-n} \ldots (\beta_m)_{k-n}} \frac{a_k z^k}{(k-n)!}. \end{aligned} \tag{1.2}$$

For brevity, (1.2) is written as

$$\widetilde{H}_n^{l,m}[\beta_1] f(z) := \widetilde{H}_n^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z). \tag{1.3}$$

Various authors have used (1.2) by means of the notation $H_n^{l,m}[\alpha_1]$, where $H_n^{l,m}$ and the parameter $\alpha_1$ in the numerator satisfy the relation

$$\alpha_1 H_n^{l,m}[\alpha_1 + 1] f(z) = z[H_n^{l,m}[\alpha_1] f(z)]' + (\alpha_1 - p) H_n^{l,m}[\alpha_1] f(z).$$

In this work, we deal with the operator $\widetilde{H}_n^{l,m}$ and the parameter $\beta_1$ satisfying the relation

$$\beta_1 \widetilde{H}_n^{l,m}[\beta_1] f(z) = z[\widetilde{H}_n^{l,m}[\beta_1 + 1] f(z)]' + (\beta_1 - n) \widetilde{H}_n^{l,m}[\beta_1 + 1] f(z). \tag{1.4}$$

Special cases of the Dziok-Srivastava linear operator includes the Hohlov linear operator [12], the Carlson-Shaffer linear operator [10], the Ruscheweyh derivative operator [20], the generalized Bernardi-Libera-Livingston linear integral operator ([9], [14], [15]) and the Srivastava-Owa fractional derivative operators ([18], [19]).

To state the main results, the following definitions and theorems will be required.

Denote by $\mathcal{Q}$ the set of all functions $q$ that are analytic and injective on $\overline{U} \setminus E(q)$ where

$$E(q) = \{\zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclasses of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$, $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

**Definition 1.1.** [16, Definition 2.3a, p. 27]. Let $\Omega$ be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and $n$ be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; z) \notin \Omega$ whenever $r = q(\zeta)$, $s = k\zeta q'(\zeta)$, and

$$\operatorname{Re}\left(\frac{t}{s} + 1\right) \geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In particular when $q(z) = M\frac{Mz+a}{M+\bar{a}z}$, with $M > 0$ and $|a| < M$, then $q(U) = U_M := \{w : |w| < M\}$, $q(0) = a$, $E(q) = \emptyset$ and $q \in \mathcal{Q}$. In this case, we set $\Psi_n[\Omega, M, a] := \Psi_n[\Omega, q]$, and in the special case when the set $\Omega = U_M$, the class is simply denoted by $\Psi_n[M, a]$.

**Definition 1.2.** [17, Definition 3, p. 817]. Let $\Omega$ be a set in $\mathbb{C}$, and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; \zeta) \in \Omega$ whenever $r = q(z), s = \frac{zq'(z)}{m}$, and

$$\operatorname{Re}\left(\frac{t}{s} + 1\right) \leq \frac{1}{m} \operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

$z \in U$, $\zeta \in \partial U$ and $m \geq n \geq 1$. In particular, we write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

**Theorem 1.1.** *[16, Theorem 2.3b, p. 28] Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ satisfies*

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega,$$

*then $p(z) \prec q(z)$.*

**Theorem 1.2.** *[17, Theorem 1, p. 818] Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathcal{Q}(a)$ and $\psi(p(z), zp'(z), z^2p''(z); z)$ is univalent in $U$, then*

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) : z \in U\}$$

*implies $q(z) \prec p(z)$.*

In the present investigation, among other results, the differential subordination result of Miller and Mocanu [16, Theorem 2.3b, p. 28] is extended for functions associated with the Dziok-Srivastava linear operator $\widetilde{H}_n^{l,m}[\beta_1]$. A similar problem was studied by Aghalary *et al.* [1]. Related results may be found in the works of [3, 4, 5, 6, 7, 8, 13]. Additionally, the corresponding differential superordination problem is also investigated, and several sandwich-type results are obtained. Analogous results for meromorphic functions in the class associated with the Liu-Srivastava operator, was recently discussed in [2].

## 2. Subordination Results Involving the Dziok-Srivastava Linear Operator

The following class of admissible functions will be required in the first result.

**Definition 2.1.** Let $\Omega$ be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_0 \cap \mathcal{H}[0, n]$. The class of admissible functions $\Phi_H[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + (\beta_1 + 1 - n)q(\zeta)}{\beta_1 + 1} \quad (\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}),$$

$$\operatorname{Re}\left(\frac{\beta_1(\beta_1 + 1)w + (n - \beta_1)(\beta_1 - n + 1)u}{(\beta_1 + 1)v + (n - \beta_1 - 1)u} - (2(\beta_1 - n) + 1)\right) \geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq n$.

In the particular case $q(z) = Mz, \quad M > 0$, and in view of Definition 2.1, the following definition is immediate.

**Definition 2.2.** Let $\Omega$ be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_H[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that

$$\phi\left(Me^{i\theta}, \frac{k+\beta_1+1-n}{\beta_1+1}Me^{i\theta}, \frac{L+(\beta_1-n+1)(2k+\beta_1-n)Me^{i\theta}}{\beta_1(\beta_1+1)}; z\right) \notin \Omega \tag{2.1}$$

whenever $z \in U$, $\theta \in \mathbb{R}$, $\operatorname{Re}(Le^{-i\theta}) \geq (k-1)kM$ for all real $\theta$, $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ and $k \geq n$.

In the special case $\Omega = q(U) = \{\omega : |\omega| < M\}$, the class $\Phi_H[\Omega, M]$ is simply denoted by $\Phi_H[M]$.

**Theorem 2.1.** *Let $\phi \in \Phi_H[\Omega, q]$. If $f \in \mathcal{A}_n$ satisfies*

$$\left\{\phi\left(\widetilde{H}_n^{l,m}[\beta_1+2]f(z), \widetilde{H}_n^{l,m}[\beta_1+1]f(z), \widetilde{H}_n^{l,m}[\beta_1]f(z); z\right) : z \in U\right\} \subset \Omega, \tag{2.2}$$

*then*

$$\widetilde{H}_n^{l,m}[\beta_1+2]f(z) \prec q(z), \qquad (z \in U).$$

**Proof.** Define the analytic function $p$ in $U$ by

$$p(z) := \widetilde{H}_n^{l,m}[\beta_1+2]f(z). \tag{2.3}$$

In view of the relation (1.4) and (2.3), it follows that

$$\widetilde{H}_n^{l,m}[\beta_1+1]f(z) = \frac{zp'(z)+(\beta_1+1-n)p(z)}{\beta_1+1}. \tag{2.4}$$

Further computations show that

$$\widetilde{H}_n^{l,m}[\beta_1]f(z) = \frac{z^2p''(z)+2(\beta_1-n+1)zp'(z)+(\beta_1-n)(\beta_1-n+1)p(z)}{\beta_1(\beta_1+1)}. \tag{2.5}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r,\ v = \frac{s+(\beta_1+1-n)r}{\beta_1+1},\ w = \frac{t+2(\beta_1-n+1)s+(\beta_1-n)(\beta_1-n+1)r}{\beta_1(\beta_1+1)}. \tag{2.6}$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z)$$
$$= \phi\left(r, \frac{s+(\beta_1+1-n)r}{\beta_1+1}, \frac{t+2(\beta_1-n+1)s+(\beta_1-n)(\beta_1-n+1)r}{\beta_1(\beta_1+1)}; z\right) \tag{2.7}$$

From (2.3), (2.4) and (2.5), the equation (2.7), yields

$$\psi(p(z), zp'(z), z^2p''(z); z)$$
$$= \phi\left(\widetilde{H}_n^{l,m}[\beta_1+2]f(z), \widetilde{H}_n^{l,m}[\beta_1+1]f(z), \widetilde{H}_n^{l,m}[\beta_1]f(z); z\right). \tag{2.8}$$

Hence (2.2) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

To complete the proof, it is left to show that the admissibility condition for $\phi \in \Phi_H[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.1. Note that

$$\frac{t}{s}+1 = \frac{\beta_1(\beta_1+1)w+(n-\beta_1)(\beta_1-n+1)u}{(\beta_1+1)v+(n-\beta_1-1)u} - (2(\beta_1-n)+1),$$

and hence $\psi \in \Psi_n[\Omega, q]$. By Theorem 1.1, $p(z) \prec q(z)$ or

$$\widetilde{H}_n^{l,m}[\beta_1+2]f(z) \prec q(z).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h$ of $U$ onto $\Omega$. In this case the class $\Phi_H[h(U), q]$ is written as $\Phi_H[h, q]$. The following result is an immediate consequence of Theorem 2.1.

**Theorem 2.2.** *Let $\phi \in \Phi_H[h, q]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_n$ satisfies*

$$\phi\left(\widetilde{H}_n^{l,m}[\beta_1+2]f(z), \widetilde{H}_n^{l,m}[\beta_1+1]f(z), \widetilde{H}_n^{l,m}[\beta_1]f(z); z\right) \prec h(z), \tag{2.9}$$

*then*

$$\widetilde{H}_n^{l,m}[\beta_1+2]f(z) \prec q(z).$$

The next result is an extension of Theorem 2.1 to the case where the behavior of $q$ on $\partial U$ is not known.

**Corollary 2.1.** *Let $\Omega \subset \mathbb{C}$, $q$ be univalent in $U$ and $q(0) = 0$. Let $\phi \in \Phi_H[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}_n$ and*

$$\phi\left(\widetilde{H}_n^{l,m}[\beta_1+2]f(z), \widetilde{H}_n^{l,m}[\beta_1+1]f(z), \widetilde{H}_n^{l,m}[\beta_1]f(z); z\right) \in \Omega,$$

*then*

$$\widetilde{H}_n^{l,m}[\beta_1+2]f(z) \prec q(z).$$

**Proof.** Theorem 2.1 yields $\widetilde{H}_n^{l,m}[\beta_1+2]f(z) \prec q_\rho(z)$. The result follows easily from the subordination $q_\rho(z) \prec q(z)$.

**Theorem 2.3.** *Let $h$ and $q$ be univalent in $U$, with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions:*

1. *$\phi \in \Phi_H[h, q_\rho]$ for some $\rho \in (0,1)$, or*
2. *there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_H[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

*If $f \in \mathcal{A}_n$ satisfies (2.9), then*

$$\widetilde{H}_n^{l,m}[\beta_1+2]f(z) \prec q(z).$$

**Proof.** The result is similar to the proof of Theorem 2.3d in [16, p. 30], and is omitted.

The next theorem yields the best dominant of the differential subordination (2.9).

**Theorem 2.4.** *Let $h$ be univalent in $U$, and $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), zq'(z), z^2q''(z); z\right) = h(z) \tag{2.10}$$

*has a solution $q$ with $q(0) = 0$ and satisfy one of the following conditions:*

1. *$q \in \mathcal{Q}_0$ and $\phi \in \Phi_H[h, q]$,*
2. *$q$ is univalent in $U$ and $\phi \in \Phi_H[h, q_\rho]$ for some $\rho \in (0,1)$, or*
3. *$q$ is univalent in $U$ and there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_H[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

*If $f \in \mathcal{A}_n$ satisfies (2.9), then*

$$\widetilde{H}_n^{l,m}[\beta_1+2]f(z) \prec q(z),$$

*and $q(z)$ is the best dominant.*

**Proof.** Following the same arguments as in [16, Theorem 2.3e, p. 31], the function $q$ is a dominant from Theorem 2.2 and Theorem 2.3. Since $q$ satisfies (2.10), it is also a solution of (2.9) and therefore $q$ will be dominated by all dominants. Hence $q$ is the best dominant.

**Corollary 2.2.** *Let $\phi \in \Phi_H[\Omega, M]$. If $f \in \mathcal{A}_n$ satisfies*

$$\phi\left(\widetilde{H}_n^{l,m}[\beta_1+2]f(z), \widetilde{H}_n^{l,m}[\beta_1+1]f(z), \widetilde{H}_n^{l,m}[\beta_1]f(z); z\right) \in \Omega,$$

*then*

$$\left|\widetilde{H}_n^{l,m}[\beta_1+2]f(z)\right| < M.$$

**Corollary 2.3.** *Let $\phi \in \Phi_H[M]$. If $f \in \mathcal{A}_n$ satisfies*

$$\left|\phi\left(\widetilde{H}_n^{l,m}[\beta_1+2]f(z), \widetilde{H}_n^{l,m}[\beta_1+1]f(z), \widetilde{H}_n^{l,m}[\beta_1]f(z); z\right)\right| < M,$$

*then*

$$\left|\widetilde{H}_n^{l,m}[\beta_1+2]f(z)\right| < M.$$

Taking $\phi(u, v, w; z) = v$ in Corollary 2.3 leads to the following example.

**Example 2.1.** *If $\operatorname{Re}\beta_1 \geq \frac{(n-k)}{2} - 1$, $k \geq p$ and $f \in \mathcal{A}_n$ satisfies*

$$\left|\widetilde{H}_n^{l,m}[\beta_1+1]f(z)\right| < M,$$

*then*

$$\left|\widetilde{H}_n^{l,m}[\beta_1+l]f(z)\right| < M.$$

*for $l = 2, 3, \cdots$.*

**Corollary 2.4.** *Let $M > 0$ and $-1 \neq \beta_1 \in \mathbb{C}$. If $f \in \mathcal{A}_n$ satisfies*

$$\left|\widetilde{H}_n^{l,m}[\beta_1+1]f(z) + \left(\frac{n}{\beta_1+1} - 1\right)\widetilde{H}_n^{l,m}[\beta_1+2]f(z)\right| < \frac{Mn}{|\beta_1+1|},$$

*then*

$$|\widetilde{H}_n^{l,m}[\beta_1+2]f(z)| < M.$$

**Proof.** Let $\phi(u,v,w;z) = v + \left(\frac{n}{\beta_1+1} - 1\right) u$ and $\Omega = h(U)$ where $h(z) = \frac{M}{|\beta_1+1|} z$, $M > 0$. It is enough to show that $\phi \in \Phi_H[\Omega, M]$, that is, the admissibility condition (2.1) is satisfied. This follows since

$$\begin{aligned}&\left|\phi\left(Me^{i\theta}, \frac{k+\beta_1+1-n}{\beta_1+1} Me^{i\theta}, \frac{L+(\beta_1-n+1)(2k+\beta_1-n)Me^{i\theta}}{\beta_1(\beta_1+1)}; z\right)\right| \\ &= \frac{kM}{|\beta_1+1|} \geq \frac{Mn}{|\beta_1+1|}\end{aligned}$$

for $z \in U$, $\theta \in \mathbb{R}$, $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ and $k \geq n$. From Corollary 2.2, the required result is obtained.

**Definition 2.3.** Let $\Omega$ be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_0 \cap \mathcal{H}_0$. The class of admissible functions $\Phi_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\phi(u,v,w;z) \notin \Omega$ whenever

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + \beta_1 q(\zeta)}{\beta_1+1} \quad (\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}),$$

$$\operatorname{Re}\left\{\frac{\beta_1[(\beta_1+1)w + (1-\beta_1)u]}{(\beta_1+1)v - \beta_1 u} + 1 - 2\beta_1\right\} \geq k \operatorname{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq 1$.

In the particular case $q(z) = Mz$, $M > 0$, and in view of Definition 2.3, the following definition is immediate.

**Definition 2.4.** Let $\Omega$ be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that

$$\phi\left(Me^{i\theta}, \frac{k+\beta_1}{\beta_1+1} Me^{i\theta}, \frac{L+\beta_1(2k+\beta_1-1)Me^{i\theta}}{\beta_1(\beta_1+1)}; z\right) \notin \Omega$$

whenever $z \in U$, $\theta \in \mathbb{R}$, $\operatorname{Re}(Le^{-i\theta}) \geq (k-1)kM$ for all real $\theta$, $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ and $k \geq 1$.

In the special case $\Omega = q(U) = \{\omega : |\omega| < M\}$, the class $\Phi_{H,1}[\Omega, M]$ is simply denoted by $\Phi_{H,1}[M]$.

**Theorem 2.5.** *Let* $\phi \in \Phi_{H,1}[\Omega, q]$. *If* $f \in \mathcal{A}_n$ *satisfies*

$$\left\{ \phi\left( \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1]f(z)}{z^{n-1}}; z \right) : z \in U \right\} \subset \Omega, \tag{2.11}$$

*then*

$$\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}} \prec q(z).$$

**Proof.** Define the analytic function $p$ in $U$ by

$$p(z) := \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}. \tag{2.12}$$

The relations (1.4) and (2.12) yield

$$\frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{z^{n-1}} = \frac{zp'(z) + \beta_1 p(z)}{\beta_1+1}. \tag{2.13}$$

Further computations show that

$$\frac{\widetilde{H}_n^{l,m}[\beta_1]f(z)}{z^{n-1}} = \frac{z^2p''(z) + 2\beta_1 zp'(z) + \beta_1(\beta_1-1)p(z)}{\beta_1(\beta_1+1)}. \tag{2.14}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r,\ v = \frac{s+\beta_1 r}{\beta_1+1},\ w = \frac{t+2\beta_1 s+\beta_1(\beta_1-1)r}{\beta_1(\beta_1+1)}. \tag{2.15}$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left( r, \frac{s+\beta_1 r}{\beta_1+1}, \frac{t+2\beta_1 s+\beta_1(\beta_1-1)r}{\beta_1(\beta_1+1)}; z \right). \tag{2.16}$$

From (2.12), (2.13) and (2.14), equation (2.16) leads to

$$\begin{aligned} &\psi(p(z), zp'(z), z^2p''(z); z) \\ &= \phi\left( \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1]f(z)}{z^{n-1}}; z \right). \end{aligned} \tag{2.17}$$

which by (2.11) gives

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

To complete the proof, the admissibility condition for $\phi \in \Phi_{H,1}[\Omega, q]$ is shown to be equivalent to the admissibility condition for $\psi$ as given in Definition 1.1. Note that

$$\frac{t}{s} + 1 = \frac{\beta_1[(\beta_1+1)w + (1-\beta_1)u]}{(\beta_1+1)v - \beta_1 u} + 1 - 2\beta_1,$$

and hence $\psi \in \Psi[\Omega, q]$. By Theorem 1.1, $p(z) \prec q(z)$ or

$$\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}} \prec q(z).$$

As in the previous case, if $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h$ of $U$ onto $\Omega$. In this case, the class $\Phi_{H,1}[h(U), q]$ is written as $\Phi_{H,1}[h, q]$. The following result is an immediate consequence of Theorem 2.5.

**Theorem 2.6.** *Let $\phi \in \Phi_{H,1}[h, q]$. If $f \in \mathcal{A}_n$ satisfies*

$$\phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1]f(z)}{z^{n-1}}; z\right) \prec h(z),$$

*then*

$$\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}} \prec q(z).$$

**Corollary 2.5.** *Let $\phi \in \Phi_{H,1}[\Omega, M]$. If $f \in \mathcal{A}_n$ satisfies*

$$\phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1]f(z)}{z^{n-1}}; z\right) \in \Omega,$$

*then*

$$\left|\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}\right| < M.$$

**Corollary 2.6.** *Let* $\phi \in \Phi_{H,1}[M]$. *If* $f \in \mathcal{A}_n$ *satisfies*

$$\left|\phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1]f(z)}{z^{n-1}}; z\right)\right| < M,$$

*then*

$$\left|\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}\right| < M.$$

Taking $\phi(u, v, w; z) = v$ in Corollary 2.6 leads to the following example.

**Example 2.2.** *If* $\operatorname{Re}\beta_1 \geq -1$ *and* $f \in \mathcal{A}_n$ *satisfies*

$$\left|\frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{z^{n-1}}\right| < M,$$

*then*

$$\left|\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}\right| < M.$$

**Remark 2.1.** The analogue between Corollary 2.3 and Corollary 2.6 can be obtained by choosing suitable admissible function.

**Definition 2.5.** Let $\Omega$ be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{H,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\phi(u, v, w; z) \notin \Omega$ whenever

$$u = q(\zeta),\ v = \frac{(\beta_1+1)q(\zeta)}{(\beta_1+2) - k\zeta q'(\zeta) - q(\zeta)}, \quad (\beta_1 \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}, \quad q(\zeta) \neq 0),$$

$$\operatorname{Re}\left\{\frac{(\beta+1)u}{v(\beta+2) - (\beta+1)u - vu}\left[\frac{\beta+1}{v} - \frac{\beta}{w} - 1\right] - \frac{\beta+1}{v} - 1\right\} \geq k \operatorname{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq 1$.

In the particular case $q(z) = 1 + Mz$, $M > 0$, and in view of Definition 2.5, the following definition is immediate.

**Definition 2.6.** Let $\Omega$ be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,2}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that

$$\phi\Bigg(1 + Me^{i\theta}, \frac{(\beta_1+1)(1+Me^{i\theta})}{(\beta_1+1) - Me^{i\theta}(k+1)},$$
$$\frac{\beta_1(1+Me^{i\theta})[\beta_1+1-Me^{i\theta}(k+1)]}{[\beta_1+1-Me^{i\theta}(k+1)][\beta_1-2Me^{i\theta}(k+1)] - (1+Me^{i\theta})(L+2Me^{i\theta})}; z\Bigg) \notin \Omega$$

whenever $z \in U$, $\theta \in \mathbb{R}$, $\operatorname{Re}\{\frac{[(\beta_1+1)-Me^{i\theta}(k+1)](1-kMe^{i\theta})-L(1+Me^{i\theta})}{kMe^{i\theta}(1+Me^{i\theta})}\} \geq (k+4)$ for all real $\theta$, $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ and $k \geq 1$.

In the special case $\Omega = q(U) = \{\omega : |\omega - 1| < M\}$, the class $\Phi_{H,2}[\Omega, M]$ is simply denoted by $\Phi_{H,2}[M]$.

**Theorem 2.7.** *Let $\phi \in \Phi_{H,2}[\Omega, q]$. If $f \in \mathcal{A}_n$ satisfies*

$$\left\{\phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_n^{l,m}[\beta_1]f(z)}; z\right) : z \in U\right\} \subset \Omega, \tag{2.18}$$

*then*

$$\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)} \prec q(z).$$

**Proof.** Define the analytic function $p$ in $U$ by

$$p(z) := \frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}. \tag{2.19}$$

By using (2.19), a computation shows that get

$$\frac{zp'(z)}{p(z)} := \frac{z[\widetilde{H}_n^{l,m}[\beta_1+3]f(z)]'}{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)} - \frac{z[\widetilde{H}_n^{l,m}[\beta_1+2]f(z)]'}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}. \tag{2.20}$$

In view of the relation (1.4) and (2.20), it follows that

$$\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)} = \frac{(\beta_1+1)p(z)}{\beta_1+2-zp'(z)-p(z)} \tag{2.21}$$

Further computations show that

$$\frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_n^{l,m}[\beta_1]f(z)} = \frac{\beta_1}{\frac{\beta_1-2-zp'(z)-p(z)}{p(z)} - \frac{zp'(z)}{p(z)} - \frac{[z^2p''(z)+2zp'(z)]}{\beta_1-2-zp'(z)-p(z)} - 1} \tag{2.22}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r,\ v = \frac{(\beta_1+1)r}{\beta_1+2-s-r},\ w = \frac{\beta_1}{\frac{\beta_1+2-s-r}{r} - \frac{s}{r} - \frac{(t+2s)}{\beta_1+2-s-r} - 1}. \tag{2.23}$$

Let

$$\begin{aligned}\psi(r,s,t;z) &:= \phi(u,v,w;z)\\ &= \phi\left(r, \frac{(\beta_1+1)r}{\beta_1+2-s-r}, \frac{\beta_1}{\frac{\beta_1+2-s-r}{r} - \frac{s}{r} - \frac{(t+2s)}{\beta_1+2-s-r} - 1}; z\right).\end{aligned} \tag{2.24}$$

From (2.19), (2.21) and (2.22), the equation(2.24), yields

$$\begin{aligned}&\psi(p(z), zp'(z), z^2p''(z); z)\\ &= \phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_n^{l,m}[\beta_1]f(z)}; z\right).\end{aligned} \tag{2.25}$$

Hence (2.18) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

To complete the proof, the admissibility condition for $\phi \in \Phi_{H,2}[\Omega, q]$ is shown to be equivalent to the admissibility condition for $\psi$ as given in Definition 1.1. Note that

$$\frac{t}{s} + 1 = \frac{(\beta+1)u}{v(\beta+2) - (\beta+1)u - vu}\left[\frac{\beta+1}{v} - \frac{\beta}{w} - 1\right] - \frac{\beta+1}{v} - 1,$$

and hence $\psi \in \Psi[\Omega, q]$. By Theorem 1.1, $p(z) \prec q(z)$ or

$$\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)} \prec q(z).$$

As in the previous cases, if $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h$ of $U$ onto $\Omega$. In this case the class $\Phi_{H,2}[h(U), q]$ is written as $\Phi_{H,2}[h, q]$. The following result is an immediate consequence of Theorem 2.7.

**Theorem 2.8.** *Let $\phi \in \Phi_{H,2}[h,q]$. If $f \in \mathcal{A}_n$ satisfies*

$$\phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_n^{l,m}[\beta_1]f(z)}; z\right) \prec h(z),$$

*then*

$$\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)} \prec q(z).$$

**Corollary 2.7.** *Let $\phi \in \Phi_{H,2}[\Omega, M]$. If $f \in \mathcal{A}_n$ satisfies*

$$\phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_n^{l,m}[\beta_1]f(z)}; z\right) \in \Omega,$$

*then*

$$\left|\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)} - 1\right| < M.$$

**Corollary 2.8.** *Let $\phi \in \Phi_{H,2}[M]$. If $f \in \mathcal{A}_n$ satisfies*

$$\left|\phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_n^{l,m}[\beta_1]f(z)}; z\right) - 1\right| < M,$$

*then*

$$\left|\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)} - 1\right| < M.$$

# 3. Superordination of the Dziok-Srivastava Linear Operator

The dual problem of differential subordination, that is, differential superordination of the Dziok-Srivastava linear operator is investigated in this section. For this purpose, the following class of admissible functions will be required.

**Definition 3.1.** Let $\Omega$ be a set in $\mathbb{C}$ and $q \in \mathcal{H}[0,p]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_H[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition $\phi(u,v,w;\zeta) \in \Omega$ whenever

$$u = q(z), \quad v = \frac{zq'(z) + m(\beta_1+1-n)q(z)}{m(\beta_1+1)} \quad (\beta_1 \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}),$$

$$\operatorname{Re}\left(\frac{\beta_1(\beta_1+1)w+(n-\beta_1)(\beta_1-n+1)u}{(\beta_1+1)v+(n-\beta_1-1)u}-(2(\beta_1-n)+1)\right)\le\frac{1}{m}\operatorname{Re}\left(\frac{zq''(z)}{q'(z)}+1\right),$$

$z\in U$, $\zeta\in\partial U$ and $m\ge n$.

**Theorem 3.1.** *Let* $\phi\in\Phi'_H[\Omega,q]$. *If* $f\in\mathcal{A}_n$, $\widetilde{H}_n^{l,m}[\beta_1+2]f(z)\in\mathcal{Q}_0$ *and*

$$\phi\left(\widetilde{H}_n^{l,m}[\beta_1+2]f(z),\widetilde{H}_n^{l,m}[\beta_1+1]f(z),\widetilde{H}_n^{l,m}[\beta_1]f(z);z\right)$$

*is univalent in* $U$, *then*

$$\Omega\subset\left\{\phi\left(\widetilde{H}_n^{l,m}[\beta_1+2]f(z),\widetilde{H}_n^{l,m}[\beta_1+1]f(z),\widetilde{H}_n^{l,m}[\beta_1]f(z);z\right):z\in U\right\}\tag{3.1}$$

*implies*

$$q(z)\prec\widetilde{H}_n^{l,m}[\beta_1+2]f(z).$$

**Proof.** From (2.8) and (3.1), it follows that

$$\Omega\subset\left\{\psi\left(p(z),zp'(z),z^2p''(z);z\right):z\in U\right\}.$$

From (2.6), it is clear that the admissibility condition for $\phi\in\Phi'_H[\Omega,q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.2. Hence $\psi\in\Psi'_p[\Omega,q]$, and by Theorem 1.2, $q(z)\prec p(z)$ or

$$q(z)\prec\widetilde{H}_n^{l,m}[\beta_1+2]f(z).$$

If $\Omega\ne\mathbb{C}$ is a simply connected domain, then $\Omega=h(U)$ for some conformal mapping $h$ of $U$ onto $\Omega$. In this case the class $\Phi'_H[h(U),q]$ is written as $\Phi'_H[h,q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.1.

**Theorem 3.2.** *Let* $h$ *be analytic in* $U$ *and* $\phi\in\Phi'_H[h,q]$. *If* $f\in\mathcal{A}_n$, $\widetilde{H}_n^{l,m}[\beta_1+2]f(z)\in\mathcal{Q}_0$ *and* $\phi\left(\widetilde{H}_n^{l,m}[\beta_1+2]f(z),\widetilde{H}_n^{l,m}[\beta_1+1]f(z),\widetilde{H}_n^{l,m}[\beta_1]f(z);z\right)$ *is univalent in* $U$, *then*

$$h(z)\prec\phi\left(\widetilde{H}_n^{l,m}[\beta_1+2]f(z),\widetilde{H}_n^{l,m}[\beta_1+1]f(z),\widetilde{H}_n^{l,m}[\beta_1]f(z);z\right)\tag{3.2}$$

*implies*

$$q(z)\prec\widetilde{H}_n^{l,m}[\beta_1+2]f(z).$$

Theorem 3.1 and 3.2 can only be used to obtain subordinants of differential superordination of the form (3.1) or (3.2). The following theorem proves the existence of the best subordinant of (3.2) for certain $\phi$.

**Theorem 3.3.** *Let $h$ be analytic in $U$ and $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi(q(z), zq'(z), z^2q''(z); z) = h(z)$$

*has a solution $q \in \mathcal{Q}_0$. If $\phi \in \Phi'_H[h,q]$, $f \in \mathcal{A}_n$, $\widetilde{H}_n^{l,m}[\beta_1+2]f(z) \in \mathcal{Q}_0$ and*

$$\phi\left(\widetilde{H}_n^{l,m}[\beta_1+2]f(z), \widetilde{H}_n^{l,m}[\beta_1+1]f(z), \widetilde{H}_n^{l,m}[\beta_1]f(z); z\right)$$

*is univalent in $U$, then*

$$h(z) \prec \phi\left(\widetilde{H}_n^{l,m}[\beta_1+2]f(z), \widetilde{H}_n^{l,m}[\beta_1+1]f(z), \widetilde{H}_n^{l,m}[\beta_1]f(z); z\right)$$

*implies*

$$q(z) \prec \widetilde{H}_n^{l,m}[\beta_1+2]f(z)$$

*and $q$ is the best subordinant.*

**Proof.** The result is similar to the proof of Theorem 2.4 and is therefore omitted.

Combining Theorems 2.2 and 3.2, we obtain the following sandwich-type theorem.

**Corollary 3.1.** *Let $h_1$ and $q_1$ be analytic functions in $U$, $h_2$ be univalent function in $U$, $q_2 \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_H[h_2, q_2] \cap \Phi'_H[h_1, q_1]$. If $f \in \mathcal{A}_n$, $\widetilde{H}_n^{l,m}[\beta_1+2]f(z) \in \mathcal{H}[0,p] \cap \mathcal{Q}_0$ and*

$$\phi\left(\widetilde{H}_n^{l,m}[\beta_1+2]f(z), \widetilde{H}_n^{l,m}[\beta_1+1]f(z), \widetilde{H}_n^{l,m}[\beta_1]f(z); z\right)$$

*is univalent in $U$, then*

$$h_1(z) \prec \phi\left(\widetilde{H}_n^{l,m}[\beta_1+2]f(z), \widetilde{H}_n^{l,m}[\beta_1+1]f(z), \widetilde{H}_n^{l,m}[\beta_1]f(z); z\right) \prec h_2(z),$$

*implies*

$$q_1(z) \prec \widetilde{H}_n^{l,m}[\beta_1+2]f(z) \prec q_2(z).$$

**Definition 3.2.** Let $\Omega$ be a set in $\mathbb{C}$ and $q \in \mathcal{H}_0$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z), \quad v = \frac{zq'(z) + m\beta_1 q(z)}{m(\beta_1 + 1)} \quad (\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}),$$

$$\mathrm{Re}\left(\frac{\beta_1[(\beta_1+1)w + (1-\beta_1)u]}{(\beta_1+1)v - \beta_1 u} + 1 - 2\beta_1\right) \leq \frac{1}{m}\,\mathrm{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

Next, the dual result of Theorem 2.5 for differential superordination will be given.

**Theorem 3.4.** *Let* $\phi \in \Phi'_{H,1}[\Omega, q]$. *If* $f \in \mathcal{A}_n$, $\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}} \in \mathcal{Q}_0$ *and*

$$\phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1]f(z)}{z^{n-1}}; z\right)$$

*is univalent in* $U$, *then*

$$\Omega \subset \left\{\phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1]f(z)}{z^{n-1}}; z\right) : z \in U\right\} \tag{3.3}$$

*implies*

$$q(z) \prec \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}.$$

**Proof.** From (2.17) and (3.3), it follows that

$$\Omega \subset \left\{\phi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in U\right\}$$

From (2.15), it follows that the admissibility condition for $\phi \in \Phi'_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 1.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}.$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h$ of $U$ onto $\Omega$. In this case the class $\Phi'_{H,1}[h(U), q]$ is written as $\Phi'_{H,1}[h, q]$. The following result is an immediate consequence of Theorem 3.4.

**Theorem 3.5.** *Let $q \in \mathcal{H}_0$, $h$ be analytic on $U$ and $\phi \in \Phi'_{H,1}[h, q]$. If $f \in \mathcal{A}_n$, $\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}} \in \mathcal{Q}_0$ and $\phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1]f(z)}{z^{n-1}}; z\right)$ is univalent in $U$, then*

$$h(z) \prec \phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1]f(z)}{z^{n-1}}; z\right)$$

*implies*

$$q(z) \prec \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}.$$

Combining Theorems 2.6 and 3.5, the following sandwich-type theorem is obtained.

**Corollary 3.2.** *Let $h_1$ and $q_1$ be analytic functions in $U$, $h_2$ be univalent function in $U$, $q_2 \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_{H,1}[h_2, q_2] \cap \Phi'_{H,1}[h_1, q_1]$. If $f \in \mathcal{A}_n$, $\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}} \in \mathcal{H}_0 \cap \mathcal{Q}_0$ and*

$$\phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1]f(z)}{z^{n-1}}; z\right)$$

*is univalent in $U$, then*

$$h_1(z) \prec \phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{z^{n-1}}, \frac{\widetilde{H}_n^{l,m}[\beta_1]f(z)}{z^{n-1}}; z\right) \prec h_2(z),$$

*implies*

$$q_1(z) \prec \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{z^{n-1}} \prec q_2(z).$$

**Definition 3.3.** Let $\Omega$ be a set in $\mathbb{C}$, $q(z) \neq 0$, $zq'(z) \neq 0$ and $q \in \mathcal{H}$. The class of admissible functions $\Phi'_{H,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z),\ v = \frac{m(\beta_1+1)q(z)}{m(\beta_1+2) - zq'(z) - mq(z)}, \quad (\beta_1 \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}, \quad q(z) \neq 0),$$

$$\operatorname{Re}\left(\frac{\beta_1 u(\beta_1+1)(w-1)}{w[(\beta_1+1)(v-u)+v(1-u)]} - \frac{(\beta+1)}{v} - 1\right) \leq \frac{1}{m}\operatorname{Re}\left(\frac{zq''(z)}{q'(z)}+1\right),$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

Now, the dual result of Theorem 2.7 for differential superordination will be given.

**Theorem 3.6.** *Let* $\phi \in \Phi'_{H,2}[\Omega, q]$. *If* $f \in \mathcal{A}_n$, $\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)} \in \mathcal{Q}_1$ *and*

$$\phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_n^{l,m}[\beta_1]f(z)}; z\right)$$

*is univalent in* $U$, *then*

$$\Omega \subset \left\{\phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_n^{l,m}[\beta_1]f(z)}; z\right) : z \in U\right\} \tag{3.4}$$

*implies*

$$q(z) \prec \frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}.$$

**Proof.** From (2.25) and (3.4), it follows that

$$\Omega \subset \left\{\phi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in U\right\}.$$

In view of (2.23), the admissibility condition for $\phi \in \Phi'_{H,2}[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 1.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}.$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h$ of $U$ onto $\Omega$. In this case the class $\Phi'_{H,2}[h(U), q]$ is written as $\Phi'_{H,2}[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.6.

**Theorem 3.7.** *Let $q \in \mathcal{H}$, $h$ be analytic in $U$ and $\phi \in \Phi'_{H,2}[h,q]$. If $f \in \mathcal{A}_n$, $\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)} \in \mathcal{Q}_1$ and $\phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_n^{l,m}[\beta_1]f(z)}; z\right)$ is univalent in $U$, then*

$$h(z) \prec \phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_n^{l,m}[\beta_1]f(z)}; z\right)$$

*implies*

$$q(z) \prec \frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}.$$

Combining Theorems 2.8 and 3.7 yield the following sandwich-type theorem.

**Corollary 3.3.** *Let $h_1$ and $q_1$ be analytic functions in $U$, $h_2$ be univalent function in $U$, $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{H,2}[h_2,q_2] \cap \Phi'_{H,2}[h_1,q_1]$. If $f \in \mathcal{A}_n$, $\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)} \in \mathcal{H} \cap \mathcal{Q}_1$ and*

$$\phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_n^{l,m}[\beta_1]f(z)}; z\right)$$

*is univalent in $U$, then*

$$h_1(z) \prec \phi\left(\frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_n^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_n^{l,m}[\beta_1]f(z)}; z\right) \prec h_2(z),$$

*implies*

$$q_1(z) \prec \frac{\widetilde{H}_n^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_n^{l,m}[\beta_1+2]f(z)} \prec q_2(z).$$

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